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DEBT AMORTIZATION AND SIMPLE INTEREST: THE CASE OF PAYMENTS IN AN ARITHMETIC PROGRESSION

AMORTIZAÇÃO DE DÍVIDAS E JUROS SIMPLES: O CASO DE PRESTAÇÕES EM PROGRESSÃO ARITMÉTICA.

AMORTIZACIÓN DE DEUDAS E INTERESES SIMPLES: EL CASO DE PAGAMENTOS EN PROGRESIÓN ARITMÉTICA.

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ABSTRACT
With the argument that, necessarily, compound interest implies anatocism, the Brazilian Judiciary has been determining that, specially for the case of debt amortization in accordance with the so called Tabela Price, when we have constant payments, simple interest must be used. With the same determination occurring in the case of the Constant Amortization Scheme, when the payments follow arithmetic progressions. However, as simple interest lacks the property of time subdivision, it is shown that, as in the case of constant payments, the adoption of simple interest in the case of payments following an arithmetic progression results in amortization schemes that are financially inconsistent. In the sense that the determination of the outstanding principal in accordance with the prospective, retrospective and of recurrence methods lead to conflicting results. To this end, four different variations of the use of simple interest are numerically analyzed.
Key words: Amortization; Simple Interest; Compound Interest

RESUMO
Com base no argumento de que, necessariamente, o regime de juros compostos implique em anatocismo, o nosso Judiciário tem determinado que, especialmente no caso de amortização de empréstimos segundo a chamada Tabela Price, onde as prestações são constantes, seja adotado o regime de juros simples. O mesmo ocorrendo no caso do Sistema de Amortização Constante, quando as prestações evoluem segundo progressões aritméticas. Todavia, por não gozar da propriedade dita de cindibilidade do prazo, evidencia-se que, do mesmo modo que no caso de prestações constantes, a adoção do regime de juros simples no caso de prestações em progressão aritmética, resulta em sistemas de amortização que são financeiramente inconsistentes. No sentido de que a determinação do saldo devedor apresenta resultados conflitantes quando se consideram os métodos prospectivo, retrospectivo e de recorrência. Isto é mostrado por meio de exemplos numéricos relativos a quatro distintas variantes.
Palavras-chave: Amortização; Juros Simples; Juros Compostos

RESUMEN

Basado en el argumento que, necesariamente, el régimen de intereses compuestos implica anatocismo, el Poder Judicial Brasileño determina que, especialmente en el caso de amortización de los préstamos según la llamada *Tabela Price*, donde los pagamentos son constantes, sea adoptado el sistema de intereses simples. Lo mismo que ocurre en el caso del Sistema de Amortización Constante, cuando los pagamentos evolucionan según progresiones aritméticas. Sin embargo, por no contar de la supuesta propiedad llamada subdivisión de plazo, muestra que, de la misma manera que en el caso de los pagamentos constantes, la adopción del sistema de intereses simples en el caso de pagamentos en progresión aritmética, resulta en sistemas de amortización que financieramente son inconsistentes. En el sentido de que la determinación del saldo deudor presenta resultados contradictorios al considerar los métodos prospectivo, retrospectivo y de recurrencia. Esto es mostrado por medio de ejemplos numéricos relativos a cuatro diferentes variantes.

Palabras claves: Amortización, Intereses Simples; Intereses Compuestos.

1. INTRODUCTION

Historically, the assessment of interest against interest, a practice that in legal jargon is termed as anatocism, has been considered not only immoral, but even illegal. Furthermore, it is customary to associate its occurrence whenever the compound interest practice is applied.

However, while anatocism implies compound interest, the opposite is not always true. As argued by de Faro (2013-a), there is no anatocism in any debt amortization scheme in which, what is understood as a negative amortization, does not occur. This suggests an apparent paradox, even when the equivalence between the principal amount of a loan and the corresponding sequence of payments is established in accordance with the principles of compound interest.

Notwithstanding, the issue is plagued by controversy. Several authors, such as De-Losso, Giovannetti & Rangel (2012), Nogueira (2013), Rovina (2009) and Sandrini (2007), contend that any debt amortization scheme based on the compound interest practice implies anatocism, namely.

Such an understanding has led to several judicial sentences by Brazilian Courts to determine that the system of constant payments, popularly known as "Tabela Price," as well as the system of constant amortization (known by the acronym SAC), both based on compound interest, should be substituted by schemes based on simple interest.

Focusing on the case of constant payments, and making use of a concept of financial consistency, de Faro (2013-b) has shown that while the "Tabela Price" conforms with such concept, the same does not occur when it adopts the simple interest practice, particularly when a peculiar variant, which has been coined as "Gauss Method", is used.

The aim of this paper is to extend the analysis to the case of SAC, which is characterized by the fact that payments follow an arithmetic progression.

The paper is further organized as follows. The second section states the definition of the concept of financial consistency. This concept is founded on the proposition that any procedure used to determine the value of the outstanding debt has to produce the same result, regardless of the amortization scheme considered.

The third section formally shows that the SAC scheme is financially consistent. Besides, in order to contrast with the cases of the simple interest schemes, which will be considered, it presents a numerical example, which is termed, "standard example".

Taking into account the intrinsic characteristic of the simple interest practice, which lacks the property of independence of the so called focal date, when establishing the equivalence between the amount loaned and the corresponding sequence or payments, the fourth section considers the two focal dates which can be regarded as most common. Furthermore, besides considering a particular variant proposed by Rovina (2009), and labeled as SAC- JS, the case of short-term loans (which employs the concept of commercial or simple discount), is also examined.

In the fourth section, considering the "standard example," and following the mathematical tradition of disproving a proposition by means of a counterexample (cf. Gelbaum and Olmsted, 2003), it is also shown that the four cases presented, fail to satisfy the concept of financial consistency.

The fifth and final section presents the conclusions.

2. THE CONCEPT OF FINANCIAL CONSISTENCY

Given a loan of value F, and the periodic rate of interest i, suppose that the loan has to be amortized by a sequence of n periodic end-of-period payments. Furthermore, it is assumed that the n payments follow an arithmetic progression with common difference R. We will assume that $R \neq 0$; otherwise, we would already have the considered case of constant payments.

Denoting as P_k the payment due k periods after the date of the loan, which is taken to be time zero, we assume that:

$$P_k = P + (k-1)R \quad , \quad k = 1, 2, ..., n \tag{1}$$

where $P > 0$ is the value of the first payment.

As in the case of SAC, it will be assumed that $R = -i.F/n$. However, R can assume any value on the real line, as long as:

a) If $R<0$, as all payments should be positive, we must have $P_n = P + (n-1)R > 0 \Rightarrow R > -P/(n-1)$;

b) If $R>0$, the corresponding value of P, which is determined when the values of F, R, n and i are given, should be such that anatocism does not occur. This requirement implies that we must have $P \geq i.F$. Otherwise, the first amortization component would be negative. This would occur, for instance, if $F = 100.000$, $n = 2$, the compound interest rate i being 20% per period, and $R=101.000$. In this case, we would have $P= 19.545,45 < i.F = 20.000$.

In order to establish the concept of financial consistency, we will make use of the following definitions:

S_k - outstanding debt at time k, with k assumed to be the time immediately after the payment P_k is due, $k = 1,2,...,n$ and $S_0 \equiv F$;

A_k - amortization component of payment P_k ,

$J_k = i.S_{k-1}$, interest component of payment P_k , with

$$P_k = A_k + J_k \tag{2}$$

It is appropriate to point out that some authors, such as De-Losso, Giovannetti & Rangel (2013) and Sandrini (2007) suggest that

$$A_k = P_k/(1+i)^k \tag{2'}$$

with J_k being given by the difference between P_k and A_k .

Referring the reader to de Faro (2014), wherein a critical analysis of such suggestion is presented, we will follow the procedure of taking A_k as given by the difference between P_k and the component, more properly named parcel of due interest, $J_k = i.S_{k-1}$.

Accordingly, we will be following the European tradition, as in de Finetti (1969, p. 138), who names A_k as "part de capitaux", Kosiol (1973, p.75), who employs the expression "tilgung," and McCutcheon and Scott (1993, p.81), who adopt the term "capital repayment." On the other hand, American authors, such as Butcher and Nesbitt (1971, p.164), as well as

Kellison (1991, p.166), make use of the denominations "principal component" and "principal repayment," respectively.

We will also employ the following basic financial principles:

1- The outstanding debt at time k, is equal to the outstanding debt at time $k\text{-}1$, increased of interest at the rate i, less the payment due at time k. That is:

$$S_k = (1+i)S_{k-1} - P_k \quad , \quad k = 1, 2, ..., n \tag{3}$$

2- If there is no overdue payment, the debt is extinguished with the payment P_n. That is, $S_n = 0$.

3- Besides the determination of the outstanding debt S_k by reiterated application of relation (3), we can also adhere to any of the following three procedures:

3.1- *retrospective method*

The outstanding debt S_k is equal to the difference between the amount F that was loaned, and the sum of the amortization components already made. That is,

$$S_k = F - \sum_{\ell=1}^{k} A_\ell \quad , \quad k = 1, 2, ..., n \tag{4}$$

Obviously, it follows that $\sum_{k=1}^{n} A_k = F$.

3.2- *prospective method*

The outstanding debt S_k is equal to the present value, computed at the

interest rate i, of the remaining $n\text{-}k$ payments.

3.3 *method of recurrence*

As it follows directly pursuant to the repeated application of relation (3), the outstanding debt at time k, S_k, can be determined by the difference between the accumulated

value of F, at the rate i, for k periods, and the accumulated value, at the same rate i, of the k payments that have already been made.

A particular debt amortization scheme is said to be financially consistent, if the outstanding debt S_k can be uniquely determined by any one of the three methods above.

3. THE CASE OF THE SAC SCHEME

As it is widely known, cf. de Faro & Lachtermacher (2012, p.267), the adoption of the SAC scheme of debt amortization implies that:

$$P_k = F(i+1/n) - i.F(k-1)/n \quad , \quad k = 1, 2, ..., n \tag{5}$$

That is, the sequence of payments follows an arithmetic progression with common difference $R = -i.F/n$ and initial term $P_1 = F(i+1/n)$.

Preliminarily, it should be observed that, considering the rate i as compound interest, the equivalence between the loan amount F and the sequence of the n payments, can be easily verified, since:

$$\sum_{k=1}^{n} P_k (1+i)^{-k} = (F/n)\left\{[1+i(n+1)]\sum_{k=1}^{n}(1+i)^{-k} - i\sum_{k=1}^{n}k(1+i)^{-k}\right\}$$
$$= (F/n)\left\{[1+i(n+1)]\left[1-(1+i)^{-n}\right]/i - (1+i)\left\{\left[1-(1+i)^{-n}\right]/i - n(1+i)^{-n-1}\right\}\right\} = F \tag{6}$$

Besides, it should also be noted that, as we do not have any negative amortization (since $A_k = F/n$ for $k = 1, 2, ..., n$), we will not have the occurrence of anatocism, in the sense as defined by Houaiss (2011), provided we do not have any overdue payment.

Let us now proceed to check if the concept of financial consistency is satisfied. Considering the determination of the outstanding debt S_k, by each one of the three procedures under consideration, we have:

a) according to the retrospective method

It is easily seen that

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$$S_k = F - \sum_{j=1}^{k} A_j = F - k.F/n = F\left(1 - k/n\right) \tag{7}$$

which shows that the outstanding debt decreases linearly with time.

    b) according to the prospective method

$$S_k = \sum_{\ell=k+1}^{n} P_\ell \left(1+i\right)^{k-\ell} = \sum_{\ell=k+1}^{n} \left\{P_{k+1} + \left(\ell - k - 1\right)R\right\}\left(1+i\right)^{k-\ell}$$

$$= \left(P_{k+1} - R\right)\left[\frac{1-\left(1+i\right)^{k-n}}{i}\right] + \frac{R\left(1+i\right)}{i}\left\{\frac{1-\left(1+i\right)^{k-n}}{i} - \left(n-k\right)\left(1+i\right)^{k-n-1}\right\} \tag{8}$$

    Thus, taking into account that

$$P_{k+1} = F\left(i + 1/n\right) - i.F.k/n$$

and

$$R = -i.F/n$$

it follows that we will also have $S_k = F\left(n-k\right)/n$

    c) according to the method of recurrence

As stated above, we should have

$$S_k = F\left(1+i\right)^k - \sum_{\ell=1}^{k} P_\ell \left(1+i\right)^{k-\ell} \tag{9}$$

    To show that relation (9) implies relation (8), it is sufficient to observe that relation (6) can be written as:

$$\sum_{\ell=1}^{k} P_\ell \left(1+i\right)^{-\ell} + \sum_{\ell=k+1}^{n} P_\ell \left(1+i\right)^{-\ell} = F \tag{6'}$$

Therefore, multiplying both sides by $\left(1+i\right)^k$, it follows that

$$\sum_{\ell=1}^{k} P_\ell \left(1+i\right)^{k-\ell} = F\left(1+i\right)^k - \sum_{\ell=k+1}^{n} P_\ell \left(1+i\right)^{k-\ell} \tag{6''}$$

Consequently, we can conclude that the prospective method and the method of recurrence lead to the same value for S_k.

In summary, the SAC scheme of debt amortization is financially consistent.

As a numerical illustration, Table I shows the evolution of the outstanding debt for the case of what is called "standard example." In this case, we have F=R$ 100.000,00, $n = 5$ periods and $i = 2\%$ for period.

Taking into consideration that P_1 = R$ 22.000,00 and R = -R$ 400,00, we have:

Table I
Evolution of the Outstanding Debt in the Case of SAC[#]

k	S_k	P_k	J_k	A_k
0	100.000,00	-	-	-
1	80.000,00	22.000,00	2.000,00	20.000,00
2	60.000,00	21.600,00	1.600,00	20.000,00
3	40.000,00	21.200,00	1.200,00	20.000,00
4	20.000,00	20.800,00	800,00	20.000,00
5	0,00	20.400,00	400,00	20.000,00
Totals	-	106.000,00	6.000,00	100.000,00

[#] Values in Brazilian reais

In this particular case, as a further numerical illustration, let us consider the determination of the outstanding debt just after the date of the third payment. We have:

a) according to the retrospective method

$$S_3 = 100.000\left(1 - 3/5\right) = R\$\, 40.000,00$$

b) according to the prospective method

$$S_3 = 20.800\left(1 + 0,02\right)^{-1} + 20.400\left(1 + 0,02\right)^{-2} = R\$\, 40.000,00$$

c) according to the method of recurrence

$$S_3 = 100.000\left(1 + 0,02\right)^3 - 22.000\left(1 + 0,02\right)^2 - 21.600\left(1 + 0,02\right)$$
$$- 21.200 = R\$\, 40.000,00$$

4. POSSIBLE EFFECTS OF SIMPLE INTEREST IMPOSITION

Let us suppose now that, by a judicial imposition, the periodic interest rate i has to be of simple interest.

At this point, as a historical curiosity, it is pertinent to recall, that although simple interest had been suggested by Wilkies (1794), currently its use for debt amortization is not mentioned at all by illustrious authors in the German, French or English languages, such as Kosiol (1973), de Finetti (1969), Butcher and Nesbitt (1971), and McCutcheon and Scott (1993), or is an object of criticisms, as in Kellison (1991, p. 82-88), who points out some inherent ambiguities.

In contrast with the case of compound interest, when the selection of the focal date is arbitrary, it is necessary to specify a particular focal date if simple interest is adopted (cf.de Faro, 1969, p.33).

Thus, two different focal dates will be considered in the process of writing down the equation of equivalence between the loan F and the corresponding sequence of payments.

As in de Faro (2013-b), the two focal dates that are most significant, will be considered, namely, the date of the loan concession, time zero, and the date of the last payment, time n.

4.1 – Taking Time Zero as the Focal Date

Once time zero is specified as the focal date, which appears to be the most logical selection, it is also necessary to specify the type of discount procedure that is going to be implemented.

In practice, we have two different possibilities. Either we make use of the so called rational discount, which applies the classical simple interest formula, or we make use of commercial discount, since the latter is used for the case of short-term loans, mainly by banks.

4.1.1- The Case of Rational Discount

If the periodic rate i is taken to be of simple interest, and the values of F, n and R are given, the value of the first payment, denoted by $\bar{P}_1$, is such that:

$$F = \sum_{k=1}^{n} \frac{\bar{P}_k}{1+k.i} = \sum_{k=1}^{n} \frac{\bar{P}_1 + (k-1)R}{1+k.i} \tag{10}$$

As shown on Table II, which considers up to five payments, the analytical expression for $\bar{P}_1$ becomes increasingly complex as n is increased.

Table II

Analytical Expression for $\bar{P}_1$

n	$\bar{P}_1$
1	$F(1+i)$
2	$\left\{F(1+3i+2i^2) - R(1+i)\right\}/(2+3i)$
3	$\left\{F(1+6i+11i^2+6i^3) - R(3+10i+7i^2)\right\}/(3+12i+11i^2)$
4	$\left\{F(1+10i+35i^2+50i^3+24i^4) - R(6+40i+80i^2+46i^3)\right\}/(4+30i+70i^2+50i^3)$
5	$\left\{F(1+15i+85i^2+225i^3+274i^4+120i^5) - R(10+110i+420i^2+646i^3+326i^4)\right\}/$ $\left(5+60i+255i^2+450i^3+274i^4\right)$

On the other hand, the numerical value of $\bar{P}_1$ can be easily determined, by employing very simple recursive procedures:

If we define

$$\alpha_\ell = \sum_{k=1}^{\ell} \frac{1}{1+k.i} \tag{11}$$

and

$$\beta_\ell = \sum_{k=1}^{\ell} \frac{k-1}{1+k.i} \tag{12}$$

let

$$\alpha_\ell = \alpha_{\ell-1} + 1/(1+\ell.i) \quad , \quad \ell = 2,3,...,n \tag{13}$$

and

$$\beta_\ell = \beta_{\ell-1} + (\ell - 1)/(1 + \ell.i) \quad , \quad \ell = 2, 3, ..., n \tag{14}$$

It follows that

$$\overline{P}_1 = (F - R.\beta_n)/\alpha_n \tag{15}$$

Considering the "standard example," and fixing $R = -$ R\$ 400,00, as in the case of SAC, it is easily verified that $\overline{P}_1 =$R\$ 21.969,80.

In Table III, where the values are in Brazilian reais, and applying relation (3) for determining the value of the outstanding debt $\overline{S}_k$, we have the evolution of the corresponding values of $\overline{S}_k$, $\overline{P}_k$, $\overline{J}_k = i.\overline{S}_{k-1}$, and $\overline{A}_k = \overline{P}_k - \overline{J}_k$.

Table III

Evolution of the Outstanding Debt for the Case of Rational Discount

k	$\overline{S}_k$	$\overline{P}_k$	$\overline{J}_k$	$\overline{A}_k$
0	100.000,00	-	-	-
1	80.030,20	21.969,80	2.000,00	19.969,80
2	60.061,00	21.569,80	1.600,60	19.969,20
3	40.092,42	21.169,80	1.201,22	19.968,58
4	20.124,47	20.769,80	801,85	19.967,95
5	157,16	20.369,80	402,49	19.967,31
Totals	-	105.849,00	6.006,16	99.842,84

It should be stressed that the debt is not fully paid even after the last payment is made. The total amortization of R\$ 99.842,84 is less than the loan amount.

Besides, we have different values for the outstanding debt when we make use of the prospective and recurrence procedures. This can be seen, for instance, if we try to determine the value of $\overline{S}_3$.

a) according to the prospective method

$$\bar{S}_3 = 20.769,80 / (1+0,02) + 20.369,80 / (1+2\times0,002) = R\$39.948,90$$

b) according to the method of recurrence

$$\bar{S}_3 = 100.000(1+3\times0,02) - 21.969,80(1+2\times0,02) - 21.569,80(1+0,02)$$
$$- 21.169,80 = R\$ 39.980,41$$

Consequently, we can conclude that the imposition of simple interest, using the rational discount procedure, and taking time zero as the focal date, lead to a financially inconsistent scheme of debt amortization.

4.1.2- The Case of Commercial Discount

Considering the case of a short-term loan, observing that we must have $n < 1 / i$, if we denote by $\hat{P}_k$ the k-th payment, the adoption of the commercial discount mechanism implies that we must have:

$$F = \sum_{k=1}^{n} \hat{P}_k \left(1 - i.k\right) = \sum_{k=1}^{n} \left\{\hat{P}_1 + (k-1)R\right\}(1 - i.k) \tag{16}$$

For the determination of the value of the first payment $\hat{P}_1$, we must recall not only the expression of the sum of the n first natural numbers, given by $n(n+1)/2$, but also of the sum of their respective squares, given by $n(n+1)(2n+1)/6$.

It follows that, given F, n, i and R, we have:

$$\hat{P}_1 = 2\left\{F - n.R\left[3(n-1) - 2i\left(n^2 - 1\right)\right]\big/6\right\}\Big/\left\{n\left[2 - i(n+1)\right]\right\} \tag{17}$$

Thus, considering the case of the "standard example," and fixing, once more R=- R\$ 400,00, we infer the first payment to be $\hat{P}_1 = R\$ 22.059,57$.

Noticing that the above value is greater than the corresponding one in the case of SAC, Table IV shows the evolution of the outstanding debt $\hat{S}_k$ (values being expressed in Brazilian reais), when applying relation (3), as well as $\hat{P}_k$, $\hat{J}_k = i.\hat{S}_{k-1}$ and $\hat{A}_k = \hat{P}_k - \hat{J}_k$.

Table IV

Evolution of the Outstanding Debt in the Case of Commercial Discount

k	$\hat{S}_k$	$\hat{P}_k$	$\hat{J}_k$	$\hat{A}_k$
0	100.000,00	-	-	-
1	79.940,43	22.059,57	2.000,00	20.059,57
2	59.879,67	21.659,57	1.598,81	20.060,76
3	39.817,69	21.259,57	1.197,59	20.061,98
4	19.754,48	20.859,57	796,35	20.063,22
5	-310,00	20.459,57	395,09	20.064,48
Totals	-	106.297,85	5.987,84	100.310,01

In this case, as we have a situation where $\hat{P}_k > P_k$, for k =1,2,3,4 and 5, it follows that one has to pay more than the value of the loan.

Furthermore, considering, for instance, the determination of $\hat{S}_3$, we have:

a) according to the prospective method

$$\hat{S}_3 = 20.859,57(1-0,02) + 20.459,57(1-2\times0,02) = R\$\,40.083,57$$

b) according to the method of recurrence

$$\hat{S}_3 = 100.000(1+3\times0,02) - 22.059,57(1+2\times0,02) - 21.659,57(1+0,02)$$
$$- 21.259,57 = R\$\,39.705,72$$

However, an incongruity should be highlighted. While future values are being determined in accordance with the principle of rational discount, present values are being computed following the commercial discount procedure. Notwithstanding, the above difference persists even if future values, *N*, were determined, starting with present values, *V*, making use of the reciprocal relation *N = V/(1-i.n)*. If this is taken into account, the method of recurrence would lead to the value

$$\hat{S}_3 = 100.000/(1 - 3 \times 0,02) - 22.059,57/(1 - 2 \times 0,02)$$
$$- 21.659,57/(1 - 0,02) - 21.259,57 = R\$\,40.043,09$$

In any event, it is clear that the practice of commercial discount also yields a financially inconsistent scheme of debt amortization.

4.2 – Taking the Date of the Last Payment as the Focal Date

In this case, in which, when considering constant payments, we have what has been denominated as the "Method of Gauss" (cf. Antonick & Assunção, 2006, and Nogueira, 2013), the equation of equivalence between the loan F and the sequence of payments, P_k^*, for $k = 1, 2, ..., n$, is written as:

$$F(1 + i.n) = \sum_{k=1}^{n} P_k^* \{1 + i(n - k)\}$$

or

$$F(1 + i.n) = \sum_{k=1}^{n} \{P_1^* + (k - 1)R\}\{1 + i(n - k)\} \tag{18}$$

Therefore, given the values of F, n, i and R, and taking into account the expressions of the sum of the first n natural numbers, and of the sum of their respective squares, it follows that:

$$P_1^* = 2\{F(1 + i.n) - n.R\{3(n - 1) + i[2 + n(n - 3)]\}/6\}/\{n[2 + i(n - 1)]\} \tag{19}$$

Thus, in the case of the "standard example", and fixing once more R = - R\$ 400,00, we have P_1^* = R\$ 21.938,46.

In Table V, still making use of relation (3) for the determination of the outstanding debt S_k^*, the evolution of S_k^*, P_k^*, and of $J_k^* = i.S_{k-1}^*$ and $A_k^* = P_k^* - J_k^*$ is shown (values in Brazilian reais).

Table V

Evolution of the Outstanding Debt Taking Time n as the Focal Date

k	S_k^*	P_k^*	J_k^*	A_k^*
0	100.000,00	-	-	-
1	80.061,54	21.938,46	2.000,00	19.938,46
2	60.124,31	21.538,46	1.601,23	19.937,23
3	40.188,34	21.138,46	1.202,49	19.935,97
4	20.253,64	20.738,46	803,77	19.934,69
5	320,26	20.338,46	405,07	19.933,39
Totals	-	105.692,30	6.012,56	99.679,74

Now, as in the case when rational discount is used and time zero is considered as the focal date, the debt is not extinguished even after the last payment is made. While this fact is sufficient to conclude that we do not have a financially consistent debt amortization scheme, let us also apply the prospective method as well as the recursive method for the determination of S_3^*. Therefore:

a) according to the prospective method

$$S_3^* = 20.738,46/(1+0,02) + 20.338,46/(1+2 \times 0,02) = R\$\ 39.888,04$$

b) according to the method of recurrence

$$S_3^* = 100.000,00(1+3 \times 0,02) - 21.938,46(1+2 \times 0,02)$$
$$-21.538,46(1+0,02) - 21.138,46 = R\$\ 40.076,31$$

It is thus clear that this type of the "Method of Gauss" approach is also financially inconsistent.

4.2.1- A Variation : the SAC – JS

In Rovina (2009), a variation for the case of taking time *n* as the focal date, identified by the acronym SAC-JS, was proposed. It was stated that the aim of this variation was to

reach a compromise between the SAC scheme and the simple interest practice, which led to the suffix, JS.

Similarly to the procedure in the "Gauss Method", the "weighted index" $\ddot{I}$ was defined as

$$\ddot{I} = 3i.F \big/ \{ n(2n.i - 2i + 3) \} \tag{20}$$

in such a way, that the k-th component of interest is considered to be reached by

$$\ddot{J}_k = (n - k + 1)\ddot{I} \quad , \quad k = 1, 2, ..., n \tag{21}$$

Then, as in the case of SAC, the specification of a constant component of amortization implies that the k- th payment can be formulated as:

$$\ddot{P}_k = F/n - (n - k + 1)\ddot{I} \quad , \quad k = 1, 2, ..., n \tag{22}$$

Observing that the equation of value

$$F(1 + n.i) = \sum_{k=1}^{n} \ddot{P}_k \{ 1 + i(n - k) \} \tag{23}$$

is satisfied, it follows that the sequence of payments forms an arithmetic progression with constant difference $\ddot{R} = -\ddot{I}$.

It should be noted that, if $R = -i.F/n$, as $i.F/n > \ddot{I}$, if $n > 1$, the payments P_k^* decrease more rapidly than the payments $\ddot{P}_k$. Therefore, for given values of n and i, noticing that in both cases the equation of equivalence between F and the corresponding sequence of payments are satisfied, considering simple interest at the rate i, and n as the focal date, it follows that $P_1^* > \ddot{P}_1$ and $P_n^* < \ddot{P}_n$.

On the other hand, with regard to the SAC scheme, as $P_1 - \ddot{P}_1 = 2i^2 F(n-1) \big/ \{ 2i(n-1) + 3 \} > 0$ if $n > 1$, we have $P_1 > \ddot{P}_1$. Furthermore, as $P_n - \ddot{P}_n = F \{ 2i^2 (n-1) \} \big/ \{ n[2i(n-1) + 3] \} > 0$,we also have $P_n > \ddot{P}_n$. Thus, if $n >$

1, we can conclude that the total amount of payments is greater in the case of the SAC scheme. A numerical comparison, in terms of the values of the first payment, is presented in the Appendix.

Turning our attention to the question of financial consistency, let us consider once again the case of the "standard example."

Observing that $\ddot{I} = R\$379,746855$, Table VI, with values in terms of Brazilian reais, depicts the evolution of the outstanding debt $\ddot{S}_k = F\left(1 - k/n\right)$, as well as the values of $\ddot{P}_k, \ddot{J}_k$ and $\ddot{A}_k$.

Table VI

Evolution of the Outstanding Debt According to the SAC-JS Methodology

k	$\ddot{S}_k$	$\ddot{P}_k$	$\ddot{J}_k$	$\ddot{A}_k$
0	100.000,00	-	-	-
1	80.000,00	21.898,73	1.898,73	20.000,00
2	60.000,00	21.518,99	1.518,90	20.000,00
3	40.000,00	21.139,24	1.139,24	20.000,00
4	20.000,00	20.759,49	759,49	20.000,00
5	0,00	20.379,75	379,75	20.000,00
Totals	-	105.696,20	5.696,20	100.000,00

Apparently, the strict adherence to the SAC-JS methodology, conforms with the retrospective method for determining the outstanding debt, as it extinguishes with the last payment.

However, such a conclusion would be misleading. This is illustrated in Table VII, where we apply relation (3), considering the values of $\ddot{P}_k$, with $J_k = i.S_{k-1}$ and $A_k = \ddot{P}_k - J_k$, which values are also expressed in Brazilian reais.

Table VII

Evolution of the Outstanding Debt Considering the Basic Relation

k	S_k	$\ddot{P}_k$	J_k	A_k

0	100.000,00	-	-	-
1	80.101,27	21.898,73	2.000,00	19.898,73
2	60.184,31	21.518,99	1.602,03	19.916,96
3	40.248,75	21.139,24	1.203,69	19.935,55
4	20.294,24	20.759,49	804,98	19.954,51
5	320,37	20.379,75	405,88	19.973,87
Totals	-	105.696,20	6.016,58	99.679,62

It is evident that not only is the debt not extinguished with the last payment, but also, while $S_3 = R\$\,40.248,75$, as derived by the retrospective method, we also have:

a) S_3 as computed by the prospective method

$$S_3 = 20.759,49/(1+0,02) + 20.379,75/(1+2\times0,02) = R\$\,39.948,35$$

b) S_3 as computed by the method of recurrence

$$S_3 = 100.000(1+0,02\times3) - 21.898,73(1+2\times0,02) - 21.518,99(1+0,02)$$
$$- 21.139,24 = R\$\,40.136,71$$

Consequently, we can conclude that the SAC-JS scheme is also not financially consistent.

5. CONCLUSION

Making use of the same approach as in de Faro (2013-b), where the case of constant payments was analyzed, three distinct possibilities were considered, as well as the SAC-JS variant, all of which make use of simple interest, leading to amortization schemes that are financially inconsistent. This is in sharp contrast with the classical System of Constant Amortization (SAC).

In this last scenario, if there is no overdue payment, we not only have absence of anatocism, although based on compound interest, but the outstanding debt can also be unambiguously determined using any one of the three classical procedures: the retrospective, the prospective, and the method of recurrence. That is, as in the case of "Tabela Price," the SAC is also an amortization scheme, which is financially consistent.

In summary, one can conclude that, independent of the focal date, any amortization system based on simple interest should be avoided. The imposition of simple interest may well lead to further litigation, since the determination of the outstanding debt, a critical issue not only in the case of anticipating the liquidation of the debt, but also when making an extraordinary amortization, cannot be uniquely achieved.

However, disregarding the evidence presented herein, if the judiciary system persists in imposing the implementation of amortization schemes, such as the SAC-JS, it may lead to critical practical questions. For instance, what would happen if the liquidation of the debt occurs before the end of the contract?

In terms of constant prices, the strict adherence to the SAC-JS methodology would imply that, at the middle of the term of the contract, the outstanding debt would be equal to half the value of the loan. However, taking into account the irrefutable argument that the outstanding debt should be determined considering only the remaining payments, the debtor may well demand the use of the prospective method. That is, the outstanding debt would have to be computed at the present value, and at the imposed simple interest rate, of the remaining payments.

However, as it can be inferred from the numerical example, which was presented, the computed value would be less than half of the value of the loan. Accordingly, the SAC-JS scheme may lead the debtor to pay more than the correct value.

Considering interest rates and terms that are effectively used in practice, a very pertinent issue is the determination of the potential loss that may have been imposed to the debtors.

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Appendix

Numerical Comparison

Fixing $F = 100.000$ units of capital, and the monthly simple interest rate $i = 2\%$, the table below presents the corresponding values of the first payments $P_1, \overline{P}_1, \hat{P}_1$ and P_1^*, when $R = -2.000/n$, as well as the values of $\ddot{P}_1$ and $\ddot{I}$, when the number n of payments is increased from 1 to 360.

n	P_1	$\overline{P}_1$	$\hat{P}_1$	P_1^*	$-R$	$\ddot{P}_1$	$\ddot{I}$
1	102.000,00	102.000,00	102.040,82	102.000,00	-	102.000,00	-
2	52.000,00	51.990,29	52.041,24	51.980,20	1.000,00	50.986,24	986,84
3	35.333,33	35.316,24	35.379,63	35.298,47	666,67	33.982,68	649,35
4	27.000,00	26.976,18	27.052,63	26.951,46	500,00	25.480,77	480,77
5	22.000,00	21.969,80	22.059,57	21.938,46	400,00	20.379,75	379,75
6	18.666,67	18.630,29	18.733,57	18.592,59	333,33	16.979,17	312,50
7	16.285,71	16.243,34	16.360,25	16.199,46	285,71	14.550,26	264,55
8	14.500,00	14.451,77	14.582,42	14.401,87	250,00	12.728,66	228,66
9	13.111,11	13.057,15	13.201,65	13.001,37	222,22	11.311,91	200,80
10	12.000,00	11.940,42	12.098,88	11.878,90	200,00	10.178,57	178,57
11	11.090,91	11.025,81	11.198,35	10.958,68	181,82	9.251,34	160,43
12	10.333,33	10.262,82	10,.449,55	10.190,19	166,67	8.478,68	145,35
24	6.166,67	6.037,62	6.407,41	5.906,96	83,33	4.230,44	63,78
36	4.777,78	4.599,40	5.191,06	4.222,50	55,56	2.815,66	37,88
48	4.083,33	3.862.39	4.750,54	3.648,15	41,67	2.108,95	25,61
60	3.666,67	3.408,37	n.a	3.163,66	33,33	1.685,32	18,66
120	2.833,33	2.437,45	n.a	2.102,79	16,67	839,78	6.44
180	2.555,56	2.067,92	n.a	1.695,37	11,11	558,84	3,28
240	2.416,67	1.861,22	n.a	1.472,73	8,33	418,66	1,99
300	2.333,33	1.724,65	n.a	1.330,84	6,67	334,67	1,34
360	2.277,78	1.652,62	n.a	1.232,03	5,56	278,74	0,96

n.a= non applicable, as n > 1/0,02 = 50 months

It should be noted that, as $i.F = 2.000$ units of capital, values of the first payment that are inferior to such a limit, should not be considered. Otherwise, we would have anatocism.

Debt Amortization and Simple Interest: The Case of Payments in an Arithmetic Progression

Apêndice
Comparação Numérica

Fixando $F = 100.000$ unidades de capital, e a taxa de juros i, admitida como mensal, em 2%, a tabela abaixo apresenta os correspondentes valores das prestações iniciais $P_1, \overline{P}_1, \hat{P}_1$ e P_1^*, quando $R = -2.000/n$, bem como os valores de $\ddot{P}_1$ e de $\ddot{I}$, para prazos n que se estendem até 360 meses.

n	P_1	$\overline{P}_1$	$\hat{P}_1$	P_1^*	$-R$	$\ddot{P}_1$	$\ddot{I}$
1	102.000,00	102.000,00	102.040,82	102.000,00	-	102.000,00	-
2	52.000,00	51.990,29	52.041,24	51.980,20	1.000,00	50.986,24	986,84
3	35.333,33	35.316,24	35.379,63	35.298,47	666,67	33.982,68	649,35
4	27.000,00	26.976,18	27.052,63	26.951,46	500,00	25.480,77	480,77
5	22.000,00	21.969,80	22.059,57	21.938,46	400,00	20.379,75	379,75
6	18.666,67	18.630,29	18.733,57	18.592,59	333,33	16.979,17	312,50
7	16.285,71	16.243,34	16.360,25	16.199,46	285,71	14.550,26	264,55
8	14.500,00	14.451,77	14.582,42	14.401,87	250,00	12.728,66	228,66
9	13.111,11	13.057,15	13.201,65	13.001,37	222,22	11.311,91	200,80
10	12.000,00	11.940,42	12.098,88	11.878,90	200,00	10.178,57	178,57
11	11.090,91	11.025,81	11.198,35	10.958,68	181,82	9.251,34	160,43
12	10.333,33	10.262,82	10,.449,55	10.190,19	166,67	8.478,68	145,35
24	6.166,67	6.037,62	6.407,41	5.906,96	83,33	4.230,44	63,78
36	4.777,78	4.599,40	5.191,06	4.222,50	55,56	2.815,66	37,88
48	4.083,33	3.862.39	4.750,54	3.648,15	41,67	2.108,95	25,61
60	3.666,67	3.408,37	n.a	3.163,66	33,33	1.685,32	18,66
120	2.833,33	2.437,45	n.a	2.102,79	16,67	839,78	6.44
180	2.555,56	2.067,92	n.a	1.695,37	11,11	558,84	3,28
240	2.416,67	1.861,22	n.a	1.472,73	8,33	418,66	1,99
300	2.333,33	1.724,65	n.a	1.330,84	6,67	334,67	1,34
360	2.277,78	1.652,62	n.a	1.232,03	5,56	278,74	0,96

Não aplicável, pois $n > 1/0,02 = 50 \text{ meses}$.

Deve ser notado que como $i.F = 2.000$ unidades de capital, valores de prestação inicial inferiores a este limite para os juros devidos não são admissíveis.